Carlo Nipoti, Dipartimento di Fisica e Astronomia, Università di Bologna

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2. The gravitational two-body problem

2.1 The reduced mass

[LL]

→ Two-body problem: two interacting particles.

→ Lagrangian

$$\mathcal{L} = \frac{1}{2}m_1|\dot{\mathbf{r}}_1|^2 + \frac{1}{2}m_2|\dot{\mathbf{r}}_2|^2 - V(|\mathbf{r}_1 - \mathbf{r}_2|)$$

→ Now take the origin in the centre of mass, so $\mathbf{r}_1 m_1 + \mathbf{r}_2 m_2 = 0$, and define $\mathbf{r} \equiv \mathbf{r}_1 - \mathbf{r}_2$, so

$$\mathbf{r}_1 = \frac{m_2}{m_1 + m_2}\mathbf{r}$$

$$\mathbf{r}_2 = -\frac{m_1}{m_1 + m_2}\mathbf{r}$$

→ Substituting these in the Lagrangian, we get

$$\mathcal{L} = \frac{1}{2}\mu^*|\dot{\mathbf{r}}|^2 - V(r),$$

where

$$\mu^* \equiv \frac{m_1 m_2}{m_1 + m_2} = \frac{m_1 m_2}{M}$$

is the reduced mass, where $M \equiv m_1 + m_2$ is the total mass.

→ So the two-body problem is reduced to the problem of the motion of one particle of mass μ^* in a central field with potential energy $V(r)$.

2.2 Kepler's problem: integration of the equations of motion

[LL; R05]

→ In the case of gravitational potential

$$V(|\mathbf{r}_1 - \mathbf{r}_2|) = -\frac{Gm_1m_2}{|\mathbf{r}_1 - \mathbf{r}_2|},$$

so

$$V(r) = -\frac{Gm_1m_2}{r} = -\frac{G\mu^*M}{r}$$

→ A widely used notation in celestial mechanics is $\mu \equiv GM = G(m_1+m_2)$, where μ is called the "gravitational mass". Note that μ does not have units of mass, while μ^* and M have units of mass.

→ The problem is reduced to the motion of a particle of mass $m = \mu^*$ in a central field with potential energy $\propto 1/r$: this is known as *Kepler's problem.* Newtonian gravity: attractive. Coulomb electrostatic interaction: attractive or repulsive.

→ Let's focus on the attractive case $\implies V = -\alpha/r$ with constant $\alpha > 0$. We are describing the motion of a particle m moving in a central potential $V = -\alpha/r$.

→ In the case of the gravitational two-body problem $m = \mu^*$ (reduced mass) and $\alpha = G(m_1 + m_2)\mu^* = GM\mu^* = \mu\mu^*$

→ We have seen that for motion in a central field, the radial motion is like 1-D motion with effective potential energy

$$V_{\rm eff}(r) = -\frac{\alpha}{r} + \frac{L^2}{2mr^2},$$

→ See plot of $V_{\rm eff}$ and energy levels (See fig 10 of LL; FIG CM2.1)

→ Minimum of $V_{\rm eff}$ at $r = L^2/m\alpha$. $V_{{\rm eff},min} = -\alpha^2 m/2L^2$. When $E = V_{{\rm eff},min}$ the orbit is circular.

→ Motion is possible only when $E > V_{\rm eff}$. If $E < 0$ motion is finite. If $E > 0$ motion is infinite.

→ Path: $\phi = \phi(r)$. Take

$$\mathrm{d}\phi = \frac{L\mathrm{d}r}{r^2\sqrt{2m[E - V_{\rm eff}(r)]}} = \frac{L\mathrm{d}r}{r^2\sqrt{2m[E - V(r)] - \frac{L^2}{r^2}}}$$

and substitute $V = -\alpha/r$.

→ We get

$$\mathrm{d}\phi = \frac{L\mathrm{d}r}{r^2\sqrt{2m\left[E + \frac{\alpha}{r}\right] - \frac{L^2}{r^2}}},$$

which can be integrated analytically to obtain:

$$\phi = \arccos\frac{(L/r) - (m\alpha/L)}{\sqrt{2mE + \frac{m^2\alpha^2}{L^2}}} + \phi_0 = \arccos\frac{\frac{L^2}{m\alpha r} - 1}{\sqrt{1 + \frac{2EL^2}{m\alpha^2}}} + \phi_0,$$

with ϕ_0 constant (verified by differentiation). Note that

$$\frac{\mathrm{d}}{\mathrm{d}x}\arccos x = -\frac{1}{\sqrt{1-x^2}}$$

→ Defining $\ell \equiv L^2/m\alpha$ and $e \equiv \sqrt{1+(2EL^2/m\alpha^2)}$ we get

$$\frac{\ell}{r} = 1 + e\cos f,$$

where $f = \phi - \phi_0$ is called the *true anomaly.*

→ This is the equation of a conic section where ℓ is the semi-latus rectum and e is the eccentricity. r is the distance from one focus. ϕ_0 is such that $\phi = \phi_0$ at the pericentre (perihelion).

→ In the two-body problem each orbit is a conic section with one focus in the centre of mass (see plot of conic sections: fig. 2.4 of MD; FIG CM2.2).

→ We have seen that for motion in a central field the time dependence of the coordinates is given by

$$\mathrm{d}t = \frac{\sqrt{m}\mathrm{d}r}{\sqrt{2[E-V(r)] - \frac{L^2}{mr^2}}},$$

which for a Kepler potential can be integrated analytically (see below, for instance for elliptic orbits: Kepler's equation).

→ Depending on the sign of E (and therefore on the value of e) we distinguish:

- $E < 0$ ($e < 1$): elliptic orbits

- $E = 0$ ($e = 1$): parabolic orbits

- $E > 0$ ($e > 1$): hyperbolic orbits

2.2.1 Elliptic orbits

[LL; R05; MD]

→ $E < 0 \implies e < 1 \implies$ elliptic orbit. Brief summary of properties of the ellipse (see fig. 4.3 of R05; FIG CM2.3). S focus, S' other focus C centre, P any point on ellipse, $CA = a$, $CB = b$, $CS = ae$, $SQ = \ell$,

$$SP/PM = e < 1 \quad \text{(eccentricity)},$$

$$SP + PS' = 2a,$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$a = \ell/(1-e^2) \quad (\mathrm{Semi-major\,axis})$$

$$b = \ell/\sqrt{1-e^2} \quad (\mathrm{Semi-minor\,axis}),$$

$$b = a\sqrt{1-e^2} \qquad (I)$$

→ From the relations among e, ℓ, L and E, we get

$$a = \ell/(1-e^2) = \alpha/2|E| \qquad (II)$$

$$b = \ell/\sqrt{1-e^2} = L/\sqrt{2m|E|} \qquad (III)$$

→ *Pericentre and apocentre.* We recall that the equation for the distance r from one of the foci is

$$r = \frac{\ell}{1+e\cos\phi} = \frac{a(1-e^2)}{1+e\cos\phi},$$

where we have assumed $\phi_0 = 0$, so $f = \phi$. Therefore the apocentre ($\cos\phi = -1$) is $r_{\rm apo} = a(1+e)$ and the pericentre ($\cos\phi = 1$) is $r_{\rm peri} = a(1-e)$.

→ We have seen that for motion in a central field the sectorial velocity $\mathrm{d}A/\mathrm{d}t$ is constant (*Kepler's second law*).

→ *Kepler's third law.* Using conservation of angular momentum

$$L = mr^2\dot{\phi} = 2m\frac{\mathrm{d}A}{\mathrm{d}t} = const$$

we get period T for elliptic orbit:

$$L\mathrm{d}t = 2m\mathrm{d}A \implies TL = 2mA = 2mab\pi,$$

where $A = \pi ab$ is the area of the ellipse.

$\implies$

$$T = 2\pi a^{3/2}\sqrt{m/\alpha} = \pi\alpha\sqrt{m/2|E|^3},$$

which is *Kepler's third law* $T \propto a^{3/2}$. Note that period depends on energy only. We have used definitions of a, b and L as functions of ℓ (semi-latus rectum): $a = \ell/(1-e^2) = \alpha/2|E|$, $b = \ell/\sqrt{1-e^2} = L/\sqrt{2m|E|}$, $L = \sqrt{m\alpha\ell}$.

→ In the case of the gravitational two-body problem we have $m = \mu^*$ and $\alpha = GM\mu^*$, so

$$T^2GM = 4\pi^2a^3 \qquad or \qquad GM = n^2a^3,$$

with $n \equiv 2\pi/T$ is the *mean motion* (i.e. the mean angular velocity),

$$E = -\frac{GM\mu^*}{2a}$$

$$e = \sqrt{1+\frac{2EL^2}{G^2M^2\mu^{*3}}} = \sqrt{1-\frac{\tilde{L}^2}{GMa}}$$

where $\tilde{L} \equiv L/\mu^* = r^2\dot{\phi}$ is the modulus of the angular momentum per unit mass.

→ In the limit $m_2 \ll m_1$ (where m_1 and m_2 are the masses of the two bodies, for instance Sun and planet), then $\mu^* \approx m_2$, $M \approx m_1$, $r_1 \approx 0$, $r_2 \approx r$, so we have *Kepler's first law*: the orbit of each planet is an ellipse with the Sun in one of its foci.

→ Using the expression for the orbital energy we can relate the velocity modulus v to r and a as follows:

$$E = T + V = \frac{1}{2}\mu^* v^2 - \frac{GM\mu^*}{r} = -\frac{GM\mu^*}{2a},$$

so

$$v^2 = GM\left(\frac{2}{r} - \frac{1}{a}\right),$$

or

$$a = \left(\frac{2}{r} - \frac{v^2}{GM}\right)^{-1}.$$

Kepler's equation

→ From the time dependence of radial coordinate (see above) we have, in the case of elliptic orbit:

$$\mathrm{d}t = \frac{r\mathrm{d}r}{\sqrt{2|E|/m}\sqrt{-r^2 + \alpha r/|E| - L^2/2m|E|}},$$

→ Note that

$$\frac{L^2}{2m|E|} = b^2 = a^2(1-e^2) = a^2 - a^2e^2$$

and

$$\alpha r/|E| = 2ar, \quad \text{because} \quad \alpha = 2a|E|$$

so

$$\mathrm{d}t = \frac{r\mathrm{d}r}{\sqrt{2|E|/m}\sqrt{a^2e^2 - (r-a)^2}}.$$

→ Let us introduce the angular variable ξ, known as the *eccentric anomaly*. We substitute

$$r = a(1 - e\cos\xi),$$

$$\mathrm{d}t = \sqrt{a^2m/2|E|}(1 - e\cos\xi)\mathrm{d}\xi$$

$$t = \sqrt{ma^3/\alpha}(\xi - e\sin\xi) + const,$$

where we have used $\alpha = 2|E|a$. Note that $0 < \xi < 2\pi$: we do the calculation for $[0, \pi]$ (so $\sin\xi = \sqrt{1-\cos^2\xi}$. The calculation for $[\pi, 2\pi]$ is similar, with $\sin\xi = -\sqrt{1-\cos^2\xi}$.

→ So, for an elliptic orbit

$$r = a(1 - e\cos\xi),$$

$$t - \tau = \sqrt{ma^3/\alpha}(\xi - e\sin\xi),$$

where τ is the time of pericentric passage (because when $t = \tau$ $\xi = 0$, so $r = a(1-e) = r_{\rm peri}$). The latter equation is known as *Kepler's equation*. Here $0 \le \xi \le 2\pi$ for one period. To obtain ξ (then r) as a function of t Kepler's equation must be solved numerically. Note that $\sqrt{ma^3/\alpha} = T/2\pi$.

→ Note that often the eccentric anomaly is indicated with E, instead of ξ (see, e.g., R05, MD).

Mean anomaly, true anomaly and eccentric anomaly.

→ In Kepler's equation ξ is the *eccentric anomaly*. Kepler's equation can be written as

$$\mathcal{M} = \xi - e\sin\xi,$$

where $\mathcal{M} = n(t-\tau)$ is the *mean anomaly*, with $n = 2\pi/T$ mean motion and $T = 2\pi a^{3/2}\sqrt{m/\alpha}$ is the period.

→ The *geometric interpretation of the eccentric anomaly* ξ is given in fig. 2.7b of MD (FIG CM2.4). So:

$$x = a\cos\xi$$

$$y^2 = b^2 - \frac{b^2}{a^2}x^2 = b^2(1-\cos^2\xi) = b^2\sin^2\xi = a^2(1-e^2)\sin^2\xi$$

$$r^2 = (x-ae)^2 + y^2 = x^2 - 2aex + a^2e^2 + y^2 = a^2\cos^2\xi - 2a^2e\cos\xi + a^2e^2 + a^2\sin^2\xi - a^2e^2\sin^2\xi =$$

$$= a^2 - 2a^2e\cos\xi + a^2e^2\cos^2\xi = a^2(1-e\cos\xi)^2$$

so

$$r = a(1-e\cos\xi)$$

→ f as a function of ξ. It is useful to derive relations between the eccentric anomaly ξ and the *true anomaly* f (see (Problem 2.**??**).

→ In summary, we recall that there are three different "anomalies": *true anomaly* f (or $\phi - \phi_0 = \phi - \omega$), *mean anomaly* $\mathcal{M} = n(t-\tau)$ and eccentric anomaly ξ (see definitions above).

Problem 2.1

Derive the expressions for ξ as a function of f and f as a function of ξ. [VK]

We know that

$$r = \frac{a(1-e^2)}{1+e\cos f}$$

so

$$1 - e\cos\xi = \frac{1-e^2}{1+e\cos f}$$

$$e\cos f = \frac{1-e^2}{1-e\cos\xi} - 1$$

$$\cos f = \frac{\cos\xi - e}{1-e\cos\xi},$$

and, using $\sin f = \sqrt{1-\cos^2 f}$,

$$\sin f = \sqrt{1-e^2}\frac{\sin\xi}{1-e\cos\xi}.$$

2.3 Systems of coordinates and orbital elements

[R05, chapter 2]

2.3.1 Celestial systems of coordinates

→ See fig. 2.5 of R05 (FIG CM2.5).

→ Celestial sphere: fictitious sphere of arbitrary radius surrounding the Earth, on which the celestial bodies are projected.

→ Celestial equator: great circle obtained intersecting the plane of the Earth equator and the celestial sphere.

→ Celestial poles: intersections of Earth axis with celestial sphere

→ Celestial meridians: great circles on the celestial sphere joining the poles

→ Plane of the ecliptic: plane of the Earth orbit around the Sun

→ Ecliptic: great circle obtained intersecting the plane of the ecliptic with the celestial sphere

→ Vernal and autumnal equinoctial points: intersections between ecliptic and celestial equator. Also known as first point of Aries and Libra.

→ Vernal equinox or first point of Aries (Υ) reference point on the ecliptic and on the celestial equator

→ Angle between ecliptic and celestial equator (i.e. inclination of Earth axis w.r.t. Earth orbit) is 23°26'

Equatorial coordinates

→ Right ascension α in hours (0-24) or degrees (0-360) from Υ eastwards.

→ Declination δ. Angle along the meridian (in degrees): 0 at the equator, 90 at the north pole, -90 at the south pole.

Ecliptic coordinates

→ Ecliptic longitude λ: in degrees (0-360) or hours (0-24) from Υ eastwards. Also known as celestial longitude.

→ Ecliptic latitude β: in degrees from 0 (ecliptic) to 90° (at the North pole of the ecliptic K). Also known as celestial latitude.

2.3.2 Orbital elements

→ For simplicity it is convenient to refer to a body orbiting the Solar System, but the same formalism applies to any body orbiting another body, when a reference plane is fixed.

→ See fig. 2.6 of R05 (FIG CM2.6).

→ Position and orbit of a celestial body (e.g. planet) defined by 6 quantities called elements. Let us specialize to the case of the elliptic orbit.

→ 3 elements define the orientation of the orbit (Ω, i, ω)

→ 2 elements define size and shape of the orbit (a, e)

→ 1 element defines position of the body at a given time (τ)

→ Line of nodes: intersection between body orbital plane and ecliptic plane

→ Nodes: intersections between ecliptic and line of nodes

→ Ascending node N: when in this node the body goes from the south ecliptic hemisphere to the north ecliptic hemisphere

(Ω) Longitude of the ascending node Ω: angle from Υ to N measured eastward on the ecliptic (in degrees from 0 to 360).

(i) Inclination i: angle between body orbital plane and ecliptic plane (in degrees)

→ Line of apses: line joining the apocentre (aphelion) and pericentre (perihelion), intercepting the celestial sphere in B (projection of pericentre) and B' (projection of apocentre)

(ω) Argument of pericentre ω: angle between N and B

(a) Semi-major axis of the elliptic orbit $a = \mu\mu^*/2|E| = GM_\odot\, \mathrm{m}_{planet}/2|E|$ (size of orbit)

(e) Eccentricity of the orbit e: distance between focus and centre is ae (shape of the orbit)

(τ) Time of pericentre passage τ: epoch at which the body was at pericentre.

→ Orbital elements for a body orbiting in the Solar System: longitude of the ascending node (Ω), inclination (i), argument of perihelion (ω), semi-major axis (a), eccentricity (e), time of perihelion passage (τ)

→ When the 6 elements are given, position of body known at any time for a two-body elliptic orbit.

→ Similar orbital elements are taken for satellites of the Earth (taking the equatorial plane as reference) or for satellites of other planets (with planet's equator as reference)

→ Also used as element the "longitude of pericentre" $\varpi = \omega + \Omega$ (dog-leg angle, because Ω and ω lie in different planes if $i \neq 0$)

→ Also used $\chi = -n\tau$ (*mean anomaly at epoch*, sometimes indicated with $\mathcal{M}_0$), instead of τ (see R05 page 211). The element χ is related to the value of the mean anomaly $\mathcal{M} \equiv n(t - \tau) = nt + \chi$ at $t = 0$.

→ Also used *mean longitude at epoch* $\epsilon = \varpi - n\tau = \varpi + \chi$ (see R05 211; MD 6.8) ϵ is the value of the *mean longitude* $\lambda \equiv \mathcal{M} + \varpi = n(t-\tau) + \varpi = nt + \epsilon$ at $t = 0$.

→ R05 uses the pair (ω, χ). MD use the pair (ϖ, ϵ).

2.4 Kepler's problem in Hamiltonian mechanics

2.4.1 Kepler's problem with Hamilton-Jacobi equation: two dimensions

[VK 4.10]

→ Let us consider for simplicity the planar problem: ϕ and r are polar coordinates in the plane of the orbit. In other words, we are assuming that the reference plane of our coordinate system coincides with the plane of the orbit (inclination $i = 0$).

→ The kinetic energy is

$$T = \frac{1}{2}\mu^* \left(\dot{r}^2 + r^2\dot{\phi}^2\right)$$

→ The potential energy is

$$V = -\frac{\mu\mu^*}{r}$$

with $\mu = G(m_1 + m_2)$

→ The Lagrangian is

$$\mathcal{L} = \frac{1}{2}\mu^* \left(\dot{r}^2 + r^2\dot{\phi}^2\right) - V(r).$$

→ The generalized momenta are

$$p_r = \frac{\partial \mathcal{L}}{\partial \dot{r}} = \mu^* \dot{r},$$

$$p_\phi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \mu^* r^2 \dot{\phi}.$$

→ The Hamiltonian is

$$\mathcal{H} = \frac{1}{2}\mu^*(\dot{r}^2 + r^2\dot{\phi}^2) - \frac{\mu\mu^*}{r},$$

$$\mathcal{H} = \frac{1}{2\mu^*}\left(p_r^2 + \frac{p_\phi^2}{r^2}\right) - \frac{\mu\mu^*}{r},$$

which does not depend on ϕ (i.e., ϕ is a cyclic coordinate)

→ We want to transform $\mathbf{q} = (r, \phi)$ and $\mathbf{p} = (p_r, p_\phi)$ into a new set of canonical coordinates $(\mathbf{Q}, \mathbf{P})$, for which the Hamiltonian is zero. So we take the action S as the generating function in the form

$$S = S(\mathbf{q}, \mathbf{P}, t) = S(r, \phi, P_1, P_2, t),$$

with P_1 and P_2 new momenta, which must be constants (because the new Hamiltonian $\mathcal{H}' = 0$).

→ Let us write the H-J equation. We have $p_i = \partial S/\partial q_i$, so $p_r = \partial S/\partial r$, $p_\phi = \partial S/\partial \phi$, and the H-J equation $\mathcal{H} + \partial S/\partial t = 0$ reads

$$\frac{1}{2\mu^*}\left[\left(\frac{\partial S}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial S}{\partial \phi}\right)^2\right] - \frac{\mu\mu^*}{r} + \frac{\partial S}{\partial t} = 0.$$

→ Using the method of separation of variables, we look for a solution in the form

$$S(r,\phi,t) = S_r(r) + S_\phi(\phi) + S_t(t)$$

→ We get

$$\frac{1}{2\mu^*}\left[\left(\frac{\mathrm{d}S_r}{\mathrm{d}r}\right)^2 + \left(\frac{1}{r}\frac{\mathrm{d}S_\phi}{\mathrm{d}\phi}\right)^2\right] - \frac{\mu\mu^*}{r} = -\frac{\mathrm{d}S_t}{\mathrm{d}t}.$$

For this to be satisfied we must have

$$\frac{\mathrm{d}S_t}{\mathrm{d}t} = -\alpha_1 = const$$

$$\frac{1}{2\mu^*}\left[\left(\frac{\mathrm{d}S_r}{\mathrm{d}r}\right)^2 + \left(\frac{1}{r}\frac{\mathrm{d}S_\phi}{\mathrm{d}\phi}\right)^2\right] - \frac{\mu\mu^*}{r} = \alpha_1,$$

which can be written as

$$\left(\frac{\mathrm{d}S_\phi}{\mathrm{d}\phi}\right)^2 = r^2\left[2\mu^*\left(\alpha_1 + \frac{\mu\mu^*}{r}\right) - \left(\frac{\mathrm{d}S_r}{\mathrm{d}r}\right)^2\right].$$

For this to be true we must have

$$\frac{\mathrm{d}S_\phi}{\mathrm{d}\phi} = \alpha_2$$

$$\frac{\mathrm{d}S_r}{\mathrm{d}r} = \sqrt{2\mu^*\left(\alpha_1 + \frac{\mu\mu^*}{r}\right) - \frac{\alpha_2^2}{r^2}}.$$

→ Thus, the generating function is

$$S = -\alpha_1 t + \alpha_2\phi + \int \mathrm{d}r\sqrt{2\mu^*\left(\alpha_1 + \frac{\mu\mu^*}{r}\right) - \frac{\alpha_2^2}{r^2}}.$$

→ We take the new generalized momenta as

$$P_1 = \alpha_1, \qquad P_2 = \alpha_2$$

so the new generalized coordinates are

$$Q_1 = \beta_1 = \frac{\partial S}{\partial P_1} = \frac{\partial S}{\partial \alpha_1}$$

$$Q_2 = \beta_2 = \frac{\partial S}{\partial P_2} = \frac{\partial S}{\partial \alpha_2}$$

→ The new Hamiltonian is

$$\mathcal{H}' = \mathcal{H} + \frac{\partial S}{\partial t} = \mathcal{H} - \alpha_1 = 0,$$

so the integral of motion

$$\alpha_1 = \mathcal{H} = \mu^*\tilde{E} = E = -\frac{\mu^*\mu}{2a}$$

is the total energy.

→ We have introduced the total energy per unit mass

$$\tilde{E} = E/\mu^* = \frac{1}{2}v^2 - \frac{\mu}{r}.$$

We focus on the elliptic case, so

$$\tilde{E} = -\frac{\mu}{2a}$$

→ We also have

$$\frac{\partial S}{\partial \phi} = p_\phi,$$

so

$$\alpha_2 = p_\phi = \mu^* r^2 \dot{\phi} = L = \mu^* \tilde{L} = \mu^* \sqrt{a\mu(1-e^2)},$$

which is the modulus of the angular momentum.

→ We have introduced the angular momentum per unit mass

$$\tilde{L} \equiv r^2 \dot{\phi} = \sqrt{a\mu(1-e^2)},$$

where we have used the definition of eccentricity:

$$e \equiv \sqrt{1 + (2EL^2/\mu^{*3}\mu^2)} = \sqrt{1 - L^2/a\mu^{*2}\mu)}.$$

→ So

$$P_1 = -\frac{\mu^*\mu}{2a}$$

$$P_2 = \mu^* \sqrt{a\mu(1-e^2)}$$

→ We now derive Q_1 and Q_2:

$$Q_1 = \frac{\partial S}{\partial \alpha_1} = -t + I_1$$

where

$$I_1 = \int \frac{\mu^* \mathrm{d}r}{\sqrt{2\mu^* \left(\alpha_1 + \frac{\mu\mu^*}{r}\right) - \frac{\alpha_2^2}{r^2}}}$$

$$= \int \frac{\mu^* \mathrm{d}r}{\sqrt{2\mu^* \left(-\frac{\mu^*\mu}{2a} + \frac{\mu\mu^*}{r}\right) - \frac{\mu^{*2} a\mu(1-e^2)}{r^2}}}$$

$$= \frac{1}{\sqrt{\mu}} \int \frac{r\mathrm{d}r}{\sqrt{-\frac{r^2}{a} + 2r - a(1-e^2)}},$$

and

$$Q_2 = \frac{\partial S}{\partial \alpha_2} = \phi - \frac{\alpha_2}{\mu^*} I_2,$$

where

$$I_2 = \int \frac{\mu^* \mathrm{d}r}{r^2 \sqrt{2\mu^* \left(\alpha_1 + \frac{\mu\mu^*}{r}\right) - \frac{\alpha_2^2}{r^2}}}$$

$$= \int \frac{\mu^* \mathrm{d}r}{r^2\sqrt{2\mu^*\left(-\frac{\mu^*\mu}{2a}+\frac{\mu\mu^*}{r}\right)-\frac{\mu^{*2}a\mu(1-e^2)}{r^2}}}$$

$$= \frac{1}{\sqrt{\mu}}\int \frac{\mathrm{d}r}{r\sqrt{-\frac{r^2}{a}+2r-a(1-e^2)}}.$$

→ Integration of I_1 (see Problem 2.2) gives

$$I_1 = \frac{a^{3/2}}{\sqrt{\mu}}(\xi - e\sin\xi),$$

where ξ is the eccentric anomaly (such that $r = a(\xi - e\cos\xi)$), so

$$Q_1 = -t + I_1 = -t + \frac{a^{3/2}}{\sqrt{\mu}}(\xi - e\sin\xi) = -t + \frac{1}{n}\mathcal{M} = -t + (t-\tau) = -\tau.$$

→ Integration of I_2 (see Problem 2.3) gives

$$I_2 = \frac{f}{\sqrt{a\mu(1-e^2)}},$$

so

$$Q_2 = \phi - \frac{\alpha_2}{\mu^*}I_2 = \phi - \frac{\alpha_2}{\mu^*}\frac{f}{\sqrt{a\mu(1-e^2)}}$$

$$= \phi - \sqrt{a\mu(1-e^2)}\frac{f}{\sqrt{a\mu(1-e^2)}} = \phi - f = \phi - (\phi - \omega) = \omega$$

where we have used

$$\alpha_2 = \mu^*\sqrt{a\mu(1-e^2)}.$$

→ In summary, the new generalized coordinates are $Q_1 = \beta_1 = -\tau$ (minus the time of pericentric passage) and $Q_2 = \beta_2 = \omega$ (argument of pericentre). The new generalized momenta are $P_1 = \alpha_1 = \mu^*\tilde{E}$ (total energy) and $P_2 = \alpha_2 = \mu^*\sqrt{a\mu(1-e^2)}$ (angular momentum modulus). All of these are constants. The new Hamiltonian is $\mathcal{H}' = 0$.

→ With the above canonical transformation we have obtained 4 constant canonical coordinates. Q_1, Q_2, P_1 and P_2 are constants of motion that fully constrain the orbit. Note that only P_1 and P_2 are integrals of motion. The solution of the equations of motion, i.e. $r = r(t)$ and $\phi = \phi(r)$, is given by the equations

$$\beta_1 = \frac{\partial S}{\partial \alpha_1}, \qquad \beta_2 = \frac{\partial S}{\partial \alpha_2}.$$

Problem 2.2

Integrate I_1.

Use the change of variables

$$r = a(1 - e\cos\xi), \qquad \mathrm{d}r = ae\sin\xi \mathrm{d}\xi,$$

where ξ is the eccentric anomaly.

We have

$$\begin{aligned} I_1 &= \frac{1}{\sqrt{\mu}} \int \frac{r\mathrm{d}r}{\sqrt{-\frac{r^2}{a} + 2r - a(1-e^2)}} \\ &= \frac{1}{\sqrt{\mu}} \int \frac{a(1-e\cos\xi)ae\sin\xi\mathrm{d}\xi}{\sqrt{-a(1-e\cos\xi)^2 + 2a(1-e\cos\xi) - a(1-e^2)}} \\ &= \frac{a^{3/2}}{\sqrt{\mu}} \int \frac{(1-e\cos\xi)e\sin\xi\mathrm{d}\xi}{\sqrt{-1+2e\cos\xi - e^2\cos^2\xi + 2 - 2e\cos\xi - 1 + e^2)}} \\ &= \frac{a^{3/2}}{\sqrt{\mu}} \int \frac{(1-e\cos\xi)e\sin\xi\mathrm{d}\xi}{\sqrt{e^2\sin^2\xi}} \\ &= \frac{a^{3/2}}{\sqrt{\mu}} \int (1-e\cos\xi)\mathrm{d}\xi = \frac{a^{3/2}}{\sqrt{\mu}}(\xi - e\sin\xi). \end{aligned}$$

Problem 2.3

Integrate I_2.

Use the change of variables

$$r = a(1 - e\cos\xi), \qquad \mathrm{d}r = ae\sin\xi \mathrm{d}\xi,$$

where ξ is the eccentric anomaly.

We have

$$\begin{aligned} I_2 &= \frac{1}{\sqrt{\mu}} \int \frac{\mathrm{d}r}{r\sqrt{-r^2/a + 2r - a(1-e^2)}} \\ &= \frac{1}{\sqrt{\mu}} \int \frac{ae\sin\xi\mathrm{d}\xi}{a(1-e\cos\xi)\sqrt{-a(1-e\cos\xi)^2 + 2a(1-e\cos\xi) - a(1-e^2)}} \\ &= \frac{1}{\sqrt{a\mu}} \int \frac{e\sin\xi\mathrm{d}\xi}{(1-e\cos\xi)\sqrt{-1+2e\cos\xi - e^2\cos^2\xi + 2 - 2e\cos\xi - 1 + e^2)}} \\ &= \frac{1}{\sqrt{a\mu}} \int \frac{e\sin\xi\mathrm{d}\xi}{(1-e\cos\xi)\sqrt{e^2\sin^2\xi}} \\ &= \frac{1}{\sqrt{a\mu}} \int \frac{\mathrm{d}\xi}{1-e\cos\xi} \\ &= \frac{1}{\sqrt{a\mu(1-e^2)}} \int \frac{\sqrt{1-e^2}\mathrm{d}\xi}{1-e\cos\xi} \end{aligned}$$

We recall that the relation between true anomaly f and eccentric anomaly ξ is

$$\sin f = \sqrt{1-e^2}\frac{\sin\xi}{1-e\cos\xi},$$

so

$$\cos f \mathrm{d}f = \sqrt{1-e^2}\frac{\cos\xi(1-e\cos\xi)-e\sin^2\xi}{(1-e\cos\xi)^2}\mathrm{d}\xi,$$

$$\cos f \mathrm{d}f = \sqrt{1-e^2}\frac{\cos\xi - e}{(1-e\cos\xi)^2}\mathrm{d}\xi,$$

$$\cos f \mathrm{d}f = \sqrt{1-e^2}\frac{\cos f}{(1-e\cos\xi)}\mathrm{d}\xi,$$

because

$$\cos f = \frac{\cos\xi - e}{1-e\cos\xi},$$

so

$$\mathrm{d}f = \frac{\sqrt{1-e^2}}{(1-e\cos\xi)}\mathrm{d}\xi,$$

thus

$$I_2 = \frac{1}{\sqrt{a\mu(1-e^2)}}\int \mathrm{d}f = \frac{f}{\sqrt{a\mu(1-e^2)}}.$$

2.4.2 Kepler's problem with Hamilton-Jacobi equation: three dimensions

[VK 4.11]

→ We now consider Kepler's problem in 3D: in practice we take a system of coordinates in which the reference plane does not coincide with the orbital plane. We recall that the motion is planar: however it is often convenient to describe it in 3D (for instance, when describing the orbit of a body in the Solar System, it is useful to take as reference plane the ecliptic: see also Chapter on perturbation theory).

→ The Lagrangian in spherical coordinates is

$$\mathcal{L} = \frac{1}{2}\mu^*\left(\dot{r}^2 + r^2\dot{\vartheta}^2 + r^2\sin^2\vartheta\dot{\phi}^2\right) + \frac{\mu^*\mu}{r}.$$

Note that when $\sin\vartheta = 1$ and $\dot{\vartheta} = 1$ we reduce the problem to the 2D case.

→ The generalized momenta are

$$p_r = \frac{\partial\mathcal{L}}{\partial\dot{r}} = \mu^*\dot{r},$$

$$p_\vartheta = \frac{\partial\mathcal{L}}{\partial\dot{\vartheta}} = \mu^* r^2\dot{\vartheta},$$

$$p_\phi = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \mu^* r^2\sin^2\vartheta\dot{\phi}.$$

→ The Hamiltonian of the Kepler problem in spherical coordinates is

$$\mathcal{H} = \frac{1}{2}\mu^*(\dot{r}^2 + r^2\dot{\vartheta}^2 + r^2\sin^2\vartheta\dot{\phi}^2) - \frac{\mu^*\mu}{r}.$$

$$\mathcal{H} = \frac{1}{2\mu^*}\left(p_r^2 + \frac{p_\vartheta^2}{r^2} + \frac{p_\phi^2}{r^2\sin^2\vartheta}\right) - \frac{\mu^*\mu}{r}.$$

→ The Hamilton-Jacobi equation reads

$$\frac{1}{2\mu^*}\left[\left(\frac{\partial S}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial S}{\partial \vartheta}\right)^2 + \left(\frac{1}{r\sin\vartheta}\frac{\partial S}{\partial \phi}\right)^2\right] - \frac{\mu^*\mu}{r} + \frac{\partial S}{\partial t} = 0,$$

→ The action as generating function is in the form $S = S(\mathbf{q}, \mathbf{P}, t) = S(r, \vartheta, \phi, P_1, P_2, P_3, t)$ with $P_i = \alpha_i = const.$ Using the method of separation of variables, we look for a solution in the form

$$S(r, \vartheta, \phi, t) = S_r(r) + S_\vartheta(\vartheta) + S_\phi(\phi) + S_t(t)$$

→ We have

$$\frac{\mathrm{d}S_t}{\mathrm{d}t} = -\alpha_1$$

$$\frac{\mathrm{d}S_\phi}{\mathrm{d}\phi} = \alpha_3$$

$$\left(\frac{\mathrm{d}S_\vartheta}{\mathrm{d}\vartheta}\right)^2 + \frac{\alpha_3^2}{\sin^2\vartheta} = \alpha_2^2$$

$$\left(\frac{\mathrm{d}S_r}{\mathrm{d}r}\right)^2 + \frac{\alpha_2^2}{r^2} = 2\mu^*\left(\alpha_1 + \frac{\mu^*\mu}{r}\right)$$

→ Performing calculations analogous to the 2-D case we get the (constant) generalized coordinates

$$Q_1 = \beta_1 = -\tau, \qquad \text{(time of pericentric passage)}$$

$$Q_2 = \omega, \qquad \text{(argument of pericentre)}$$

$$Q_3 = \Omega, \qquad \text{(longitude of the ascending node)},$$

and the (constant) generalized momenta

$$P_1 = E = -\frac{\mu^*\mu}{2a}, \qquad \text{(total energy)},$$

$$P_2 = L = \mu^*\sqrt{a\mu(1-e^2)}, \qquad \text{(angular momentum modulus)},$$

$$P_3 = L_z = \mu^*\sqrt{a\mu(1-e^2)}\cos i, \qquad \text{(z-component of angular momentum)}.$$

The corresponding Hamiltonian is $\mathcal{H}' = 0$.

→ With the above canonical transformation we have obtained 6 constant canonical coordinates, i.e. 6 constants of motion (only P_1, P_2 and P_3 are integrals of motion). These integrals of motions fully constrain the orbit. The solution of the equations of motion is given by the equations

$$\beta_i = \frac{\partial S}{\partial \alpha_i}, \qquad i = 1, 2, 3.$$

→ It is also possible to obtain another set of canonical coordinates, maintaining the generalized coordinates as they are, but mass-normalizing the generalized momenta and the Hamiltonian. So, we start from Q_i, P_i and define a new set of variables $\tilde{q}_i = Q_i$, $\tilde{p}_i = P_i/\mu^*$. The equations of motion keep the canonical form with the Hamiltonian $\tilde{\mathcal{H}} = \mathcal{H}'/\mu^*$ (i.e. this is a canonical transformation; see G09). This can be seen also by noting that

$$\dot{P}_i = -\frac{\partial \mathcal{H}'}{\partial Q} \implies \frac{\dot{P}_i}{\mu^*} = -\frac{\partial(\mathcal{H}'/\mu^*)}{\partial Q_i} \implies \dot{\tilde{p}}_i = -\frac{\partial \tilde{\mathcal{H}}}{\partial \tilde{q}_i}$$

and

$$\dot{Q}_i = \frac{\partial \mathcal{H}'}{\partial P_i} = \frac{\partial(\mathcal{H}'/\mu^*)}{\partial P_i/\mu^*} \implies \dot{\tilde{q}}_i = \frac{\partial \tilde{\mathcal{H}}}{\partial \tilde{p}_i}.$$

→ The (constant) mass-normalized canonical coordinates are

$$\tilde{\beta}_1 = \tilde{q}_1 = -\tau, \qquad \tilde{\beta}_2 = \tilde{q}_2 = \omega, \qquad \tilde{\beta}_3 = \tilde{q}_3 = \Omega$$

$$\tilde{\alpha}_1 = \tilde{p}_1 = -\frac{\mu}{2a}, \qquad \tilde{\alpha}_2 = \tilde{p}_2 = \sqrt{a\mu(1-e^2)}, \qquad \tilde{\alpha}_3 = \tilde{p}_3 = \sqrt{a\mu(1-e^2)}\cos i.$$

The mass-normalized Hamiltonian is $\tilde{\mathcal{H}}' = \mathcal{H}'/\mu^* = 0$.

2.4.3 Angle-action variables for Kepler's problem

[GPS; BT08; G09]

→ We now consider again Kepler's problem in 3D, but now we derive the solution in terms of angle-action variables.

→ The Hamiltonian of the Kepler problem in spherical coordinates is

$$\mathcal{H} = \frac{1}{2\mu^*}\left(p_r^2 + \frac{p_\vartheta^2}{r^2} + \frac{p_\phi^2}{r^2 \sin^2\vartheta}\right) - \frac{\mu^*\mu}{r}.$$

→ The mass-normalized Hamiltonian is

$$\tilde{\mathcal{H}} = \frac{1}{2\mu^{*2}}\left(p_r^2 + \frac{p_\vartheta^2}{r^2} + \frac{p_\phi^2}{r^2 \sin^2\vartheta}\right) - \frac{\mu}{r} = \frac{1}{2}\left(\tilde{p}_r^2 + \frac{\tilde{p}_\vartheta^2}{r^2} + \frac{\tilde{p}_\phi^2}{r^2 \sin^2\vartheta}\right) - \frac{\mu}{r},$$

where $\tilde{p}_r = p_r/\mu^*$, $\tilde{p}_\vartheta = p_\vartheta/\mu^*$ and $\tilde{p}_\phi = p_\phi/\mu^*$.

→ In terms of Hamilton characteristic function W, the Hamilton-Jacobi equation reads

$$\frac{1}{2}\left[\left(\frac{\partial W}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial W}{\partial \vartheta}\right)^2 + \left(\frac{1}{r\sin\vartheta}\frac{\partial W}{\partial \phi}\right)^2\right] - \frac{\mu}{r} = \tilde{E},$$

→ Using the method of separation of variables, we look for a solution in the form

$$W(r,\vartheta,\phi) = W_r(r) + W_\vartheta(\vartheta) + W_\phi(\phi)$$

→ We must have

$$\frac{\partial W_\phi}{\partial \phi} = \tilde{\alpha}_3$$

$$\left(\frac{\partial W_\vartheta}{\partial \vartheta}\right)^2 + \frac{\tilde{\alpha}_3^2}{\sin^2\vartheta} = \alpha_\vartheta^2$$

$$\left(\frac{\partial W_r}{\partial r}\right)^2 + \frac{\tilde{\alpha}_2^2}{r^2} - 2\frac{\mu}{r} = 2\tilde{E},$$

with $\tilde{\alpha}_2$ and $\tilde{\alpha}_3$ constant.

→ The above can be written as

$$p_\phi^2 = \tilde{\alpha}_3^2 = \tilde{L}_z^2$$

$$p_\vartheta^2 = \tilde{\alpha}_2^2 - \frac{\tilde{\alpha}_3^2}{\sin^2\vartheta} = \tilde{L}^2 - \frac{\tilde{L}_z^2}{\sin^2\vartheta},$$

$$p_r^2 = 2\tilde{E} + 2\frac{\mu}{r} - \frac{\tilde{\alpha}_2^2}{r^2}.$$

→ The (mass-normalized) action variables are

$$J_\phi = \frac{1}{2\pi}\oint \tilde{p}_\phi \mathrm{d}\phi = \frac{1}{2\pi}\oint \frac{\partial W_\phi}{\partial \phi}\mathrm{d}\phi,$$

$$J_\vartheta = \frac{1}{2\pi}\oint \tilde{p}_\vartheta \mathrm{d}\vartheta = \frac{1}{2\pi}\oint \frac{\partial W_\vartheta}{\partial \vartheta}\mathrm{d}\vartheta,$$

$$J_r = \frac{1}{2\pi}\oint \tilde{p}_r \mathrm{d}r = \frac{1}{2\pi}\oint \frac{\partial W_r}{\partial r}\mathrm{d}r.$$

→ So

$$J_\phi = \frac{1}{2\pi}\oint \tilde{L}_z \mathrm{d}\phi,$$

$$J_\vartheta = \frac{1}{2\pi}\oint \sqrt{\tilde{L}^2 - \frac{\tilde{L}_z^2}{\sin^2\vartheta}}\mathrm{d}\vartheta$$

$$J_r = \frac{1}{2\pi}\oint \sqrt{2\tilde{E} + \frac{2\mu}{r} - \frac{\tilde{L}^2}{r^2}}\mathrm{d}r,$$

→ The first integral gives

$$J_\phi = \frac{1}{2\pi}\oint \tilde{L}_z \mathrm{d}\phi = \tilde{L}_z.$$

→ To compute the second integral, note that the integral over ϑ along the closed path γ_ϑ equals four times the integral between $\vartheta = \pi/2$ and $\vartheta = \vartheta_{min}$ where $\vartheta_{min} < \pi/2$ is such that $\sin\vartheta_{min} = |\cos i|$, where i is the inclination. So $\sin\vartheta_{min} = |L_z|/L = |\tilde{L}_z|/\tilde{L}$. The integral becomes

$$J_\vartheta = \frac{1}{2\pi}\oint\sqrt{\tilde{L}^2 - \frac{\tilde{L}_z^2}{\sin^2\vartheta}} = \frac{2}{\pi}\int_{\vartheta_{min}}^{\pi/2}\sqrt{\tilde{L}^2 - \frac{\tilde{L}_z^2}{\sin^2\vartheta}},$$

which can be computed analytically (see, e.g., GPS or G09), eventually giving $J_\vartheta = \tilde{L} - |\tilde{L}_z|$. Therefore $\tilde{L} = J_\vartheta + |J_\phi|$, because $J_\phi = \tilde{L}_z$.

→ The radial integral can be written as twice the integral between r_{min} and r_{max}, so

$$J_r = \frac{1}{2\pi}\oint\sqrt{2\tilde{E} + \frac{2\mu}{r} - \frac{(J_\vartheta + |J_\phi|)^2}{r^2}}\mathrm{d}r = \frac{1}{\pi}\int_{r_{min}}^{r_{max}}\sqrt{2\tilde{E} + \frac{2\mu}{r} - \frac{(J_\vartheta + \|Jphi|)^2}{r^2}}\mathrm{d}r.$$

→ It can be shown that integration of the above gives (see GPS or G09)

$$J_r = -(J_\vartheta + |J_\phi|) + \frac{\mu}{\sqrt{-2\tilde{E}}},$$

so

$$\tilde{\mathcal{H}} = \tilde{E} = -\frac{\mu^2}{2(J_r + J_\vartheta + |J_\phi|)^2}.$$

→ The angle variables θ_r, θ_ϑ and θ_ϕ can be computed from

$$\theta_i = \frac{\partial W}{\partial J_i}.$$

However we are most interested to compute the associated frequencies, which are given by

$$\Omega_i = \dot{\theta}_i = \frac{\partial\tilde{\mathcal{H}}}{\partial J_i}.$$

Given the degenerate form of the Hamiltonian all frequencies are equal:

$$\Omega_r = \Omega_\vartheta = \frac{\mu^2}{(J_r + J_\vartheta + |J_\phi|)^3},$$

and $\Omega_\phi = \mathrm{sgn}(J_\phi)\Omega_r$.

2.4.4 Delaunay's variables

[GPS; BT08; VK]

→ The degenerate form of the Hamiltonian for the action variables J_r, J_ϑ, J_ϕ suggests to look for a new set of angle-action variables (J_a, J_b, J_c) such that one of the actions is $J_r + J_\vartheta + |J_\phi|$. These new angle-action variables (known as Delaunay variables) are obtained by the canonical transformation with generating function

$$F(\theta_r, \theta_\vartheta, \theta_\phi, J_a, J_b, J_c) = \theta_\phi J_a + \theta_\vartheta(J_b - |J_a|) + \theta_r(J_c - J_b).$$

→ The new actions can be obtained from

$$J_r = \frac{\partial F}{\partial \theta_r} = J_c - J_b$$

$$J_\vartheta = \frac{\partial F}{\partial \theta_\vartheta} = J_b - |J_a|$$

$$J_\phi = \frac{\partial F}{\partial \theta_\phi} = J_a$$

so

$$J_a = J_\phi, \qquad J_b = J_\vartheta + |J_\phi|, \qquad J_c = J_\vartheta + J_r + |J_\phi|$$

→ The new (Delaunay) Hamiltonian is $H_D = -\mu^2/2J_c^2$.

→ The new angles are

$$\theta_a = \frac{\partial F}{\partial J_a} = \theta_\phi - \mathrm{sgn}(J_a)\theta_\vartheta,$$

$$\theta_b = \frac{\partial F}{\partial J_b} = \theta_\vartheta - \theta_r,$$

$$\theta_c = \frac{\partial F}{\partial J_c} = \theta_r.$$

→ The new frequencies are

$$\Omega_i = \dot{\theta}_i = \frac{\partial H_D}{\partial J_i}, \qquad (i = a, b, c)$$

so

$$\Omega_a = \Omega_b = 0, \qquad \Omega_c = \frac{\mu^2}{J_c^3}.$$

→ The actions can be expressed in terms of the orbital elements:

$$J_a = J_\phi = \tilde{L}_z = \sqrt{\mu a(1-e^2)}\cos i,$$

$$J_b = J_\vartheta + |J_\phi| = \tilde{L} = \sqrt{\mu a(1-e^2)},$$

$$J_c^2 = -\mu^2/2\tilde{E} = \sqrt{\mu a}$$

.

→ Finally we want to express the Delaunay angles in terms of the orbital elements. We know θ_a, θ_b, θ_c as functions of θ_r, θ_ϑ, θ_ϕ, which in principle can be computed by integrating the equations $\theta_i = \partial W_i/\partial J_i$ $(i = r, \vartheta, \phi)$. However we can compute directly θ_a, θ_b, θ_c exploiting our knowledge that a set of (constant) mass-normalized canonical coordinates is

$$\tilde{q}_1 = -\tau, \qquad \tilde{q}_2 = \omega, \qquad \tilde{q}_3 = \Omega$$

$$\tilde{p}_1 = -\frac{\mu}{2a}, \qquad \tilde{p}_2 = \sqrt{a\mu(1-e^2)}, \qquad \tilde{p}_3 = \sqrt{a\mu(1-e^2)}\cos i.$$

→ We just need to perform a canonical transformation from ($\tilde{q}_1$,$\tilde{q}_2$,$\tilde{q}_3$,$\tilde{p}_1$,$\tilde{p}_2$,$\tilde{p}_3$) to $(J_a, J_b, J_c, \theta_a, \theta_b, \theta_c)$. The generating function is

$$F(\tilde{q}_1, \tilde{q}_2, \tilde{q}_3, J_a, J_b, J_c, t) = \left(nJ_c - \frac{3\mu}{2a}\right)(t + \tilde{q}_1) + \tilde{q}_2 J_b + \tilde{q}_3 J_a,$$

→ We have

$$\theta_a = \frac{\partial F}{\partial J_a} = \tilde{q}_3 = \Omega$$

$$\theta_b = \frac{\partial F}{\partial J_b} = \tilde{q}_2 = \omega$$

$$\theta_c = \frac{\partial F}{\partial J_c} = n(t + \tilde{q}_1) = n(t - \tau) = \mathcal{M}$$

$$\tilde{p}_1 = \frac{\partial F}{\partial \tilde{q}_1} = nJ_c - \frac{3\mu}{2a} = \frac{\mu^{1/2}}{a^{3/2}}\sqrt{\mu a} - \frac{3\mu}{2a} = -\frac{\mu}{2a}$$

$$\tilde{p}_2 = \frac{\partial F}{\partial \tilde{q}_2} = J_b = \sqrt{\mu a(1 - e^2)},$$

$$\tilde{p}_3 = \frac{\partial F}{\partial \tilde{q}_3} = J_a = \sqrt{\mu a(1 - e^2)}\cos i,$$

→ In summary, Delaunay angle-action variables are

$$\theta_a = \Omega, \qquad \theta_b = \omega, \qquad \theta_c = n(t - \tau) = \mathcal{M}$$

$$J_a = \sqrt{a\mu(1 - e^2)}\cos i, \qquad J_b = \sqrt{a\mu(1 - e^2)}, \qquad J_c = \sqrt{a\mu},$$

and the corresponding Hamiltonian is $H_D = -\mu^2/2J_c^2$.

→ Five of the Delaunay's variables are constants, but θ_c (mean anomaly) is not constant and varies linearly with time.

→ Note that traditionally the Delaunay actions are indicated with $L = J_c$, $G = J_b$, $H = J_a$, and the corresponding angles $l = \theta_c$, $g = \theta_b$, $h = \theta_a$.

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